Technical Report No. 5

ON SOLVING ELASTIC WAVEGUIDE PROBLEMS
INVOLVING NON-MIXED EDGE CONDITIONS

by

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ABSTRACT

Within the framework of the "exact" linear theory an important class of wave propagation problems in elastic waveguides, involving non-mixed edge conditions (like stress or displacement), have remained unsolved. Basically, this is because known separation methods (classical or integral transforms) do not "ask" in a natural way for the given edge information. A means for solving some problems in this class, focused on the semi-infinite plate, as an example, is presented here. In the method a Laplace transform is used on the propagation coordinate, say x. Exploitation of the boundedness condition on the solution, at $x \to \infty$, generates two coupled integral equations for the edge unknowns (displacements and strains), which depend, parametrically, on those complex wave number roots of the governing Rayleigh-Lamb frequency equation representing unbounded waves. Solution of these equations determines the transformed solution of the problem, which can be inverted through known techniques. Excitation of a plate with a built-in edge is treated as an example.

§1. INTRODUCTION

Attempts at solving waveguide problems based on the equations of motion from linear elasticity theory, and involving non-mixed edge conditions, date back to Pochhammer's classical work [1][1] in 1876 on the frequency equation for the infinite circular cylindrical rod. As Love [2] shows, attempts to use this theory to treat the free vibration of the finite length, homogeneous,

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[1] Numbers in brackets designate references at the end of the report.

isotropic rod, with a separation technique, lead to a solution in which the normal stress on the rod ends vanishes, but not the shear stress. Only an approximate solution could be written based on the argument that if the rod was long and thin, with the shear stress zero on the cylindrical surface, it is also approximately zero on the ends.

Similar difficulties are evidenced in the modern work done in elastic waveguides. For example, the integral transform technique given by Folk, Fox, Shook, and Curtis [3] for solving semi-infinite elastic rod problems, involving transient end load inputs, are restricted to mixed-end conditions, i. e. they treat, for instance, a case of suddenly applied normal stress at the rod end under radial (displacement) constraint. Simply put, those elastic waveguide problems, that have been solved by a direct integral transform method, have been geared to the mixed-end (or edge) conditions, since these are what the transforms "asked" for.

In problems involving non-mixed end (or edge) conditions, i. e. the stresses or displacements, very little information has been obtained, basically because direct separation methods fail to yield solutions. A means for solving some problems in this class, focused on the semi-infinite plate, as an example, is presented here. In the method a Laplace transform, parameter s, is used on the propagation coordinate x. To insure that a solution be bounded at $x \to \infty$, residues of poles in the transform of the solution, occurring in the right half s-plane and corresponding to complex wave number roots of the governing Rayleigh-Lamb frequency equation, are set equal to zero. This generates two coupled integral equations for the edge unknowns (displacements and strains) which depend, parametrically, on these complex wave number roots. Solution of these equations determines

the transformed solution of the problem, which can be inverted through known techniques[2]. Excitation of a plate with a built-in edge is treated as an example.

§2. GENERAL METHOD

Formal Solution

Consider the semi-infinite elastic plate of thickness 2h sketched in Fig. 1. If we let u, v, w be the displacements in the x, y, and z directions, respectively, and assume plane strain such that $w = \frac{\partial}{\partial z} = 0$, the governing displacement equations of motion from linear elasticity theory, for the homogeneous, isotropic plate, can be written as

$$\left.\begin{array}{l} c_d^2 u_{xx}(x,y,t) + (c_d^2 - c_s^2)v_{xy}(x,y,t) + c_s^2 u_{yy}(x,y,t) = u_{tt}(x,y,t) \\[3mm] c_s^2 v_{xx}(x,y,t) + (c_d^2 - c_s^2)u_{xy}(x,y,t) + c_d^2 v_{yy}(x,y,t) = v_{tt}(x,y,t) \end{array}\right\} \quad (1)$$

where $c_d^2 = (\lambda+2\mu)/\rho$, and $c_s^2 = \mu/\rho$ are, respectively, the dilatational and equivoluminal body wave speeds, λ and μ being Lamé's constants. Corresponding stress-strain relations are

$$\left.\begin{array}{l} \dfrac{\sigma_x(x,y,t)}{\lambda+2\mu} = u_x(x,y,t) + \left(\dfrac{k^2-2}{k^2}\right)v_y(x,y,t) \; , \qquad k^2 = \dfrac{c_d^2}{c_s^2} \\[5mm] \dfrac{\sigma_y(x,y,t)}{\lambda+2\mu} = \left(\dfrac{k^2-2}{k^2}\right)u_x(x,y,t) + v_y(x,y,t) \\[5mm] \dfrac{\sigma_{xy}(x,y,t)}{\mu} = v_x(x,y,t) + u_y(x,y,t) \; , \qquad \text{and} \;\; \sigma_z = \nu(\sigma_x + \sigma_y) \end{array}\right\} \quad (2)$$

[2] The method developed here is an extension of one developed by M. Picone that the author learned of through private communications with Professors A. Ghizzetti and W. Cross of the Instituto Nazionale Per Le Applicazioni Del Calcolo, Rome, Italy. Picone was interested in finite domain problems, hence used a finite Laplace transform and the entirety of this transform in the s-plane. Evidently this work was not published, but Picone's related work on the finite Laplace transform [4] was published in 1939.

where ν is Poisson's ratio. Subscripts in this work, when associated with displacement, indicate differentiation, but when associated with stress identify the component in the usual way. Initial conditions are taken as

$$u(x, y, 0) = u_t(x, y, 0) = v(x, y, 0) = v_t(x, y, 0) = 0 \tag{3}$$

and conditions at $x \to \infty$ as

$$\lim_{x \to \infty} \left\{ \begin{array}{ll} u, & u_x, \quad \text{etc.} \\ v, & v_x, \quad \text{etc.} \end{array} \right\} = 0 \tag{4}$$

Now introducing the one sided Laplace transforms on x and t, the transforms of (1), using (3), are

$$\widetilde{u}_{yy}(s, y, p) + (k^2 - 1)s\widetilde{v}_y(s, y, p) + (k^2 s^2 - k_s^2)\widetilde{u}(s, y, p)$$

$$= k^2 [s\overline{u}(0, y, p) + \overline{u}_x(0, y, p)] + (k^2 - 1)\overline{v}_y(0, y, p) = f(y)$$

$$\widetilde{v}_{yy}(s, y, p) + \left(\frac{k^2 - 1}{k^2}\right)s\widetilde{u}_y(s, y, p) + \left(\frac{s^2 - k_s^2}{k^2}\right)\widetilde{v}(s, y, p)$$

$$= \frac{1}{k^2}[s\overline{v}(0, y, p) + \overline{v}_x(0, y, p)] + \left(\frac{k^2 - 1}{k^2}\right)\overline{u}_y(0, y, p) = g(y) \tag{5}$$

where $k_d^2 = p^2/c_d^2$ and $k_s^2 = p^2/c_s^2$, and the bar and tilda over quantities indicate the time and space transforms, respectively. It is important to note that f(y) and g(y) are composed of the edge unknowns $\overline{u}$, $\overline{u}_x$, and $\overline{v}_y$, and $\overline{v}$, $\overline{v}_x$, and $\overline{u}_y$, respectively, these quantities being the time transformed displacements and strains. Likewise, the transforms of (2) are

$$\frac{\widetilde{\sigma}_x(s, y, p)}{\lambda + 2\mu} = s\widetilde{u}(s, y, p) - \overline{u}(0, y, p) + \left(\frac{k^2 - 2}{k^2}\right)\widetilde{v}_y(s, y, p)$$

$$\frac{\widetilde{\sigma}_y(s, y, p)}{\lambda + 2\mu} = \left(\frac{k^2 - 2}{k^2}\right)[s\widetilde{u}(s, y, p) - \overline{u}(0, y, p)] + \widetilde{v}_y(s, y, p) \tag{6}$$

$$\frac{\widetilde{\sigma}_{xy}'(s,y,p)}{\mu} = \widetilde{u}_y(s,y,p) + s\widetilde{v}(s,y,p) - \overline{v}(0,y,p)$$

$$\widetilde{\sigma}_z(s,y,p) = \nu[\widetilde{\sigma}_x(s,y,p) + \widetilde{\sigma}_y(s,y,p)]$$

$$(6)$$

The solution of the fourth order system of homogeneous equations in (5) can be found, in the usual manner, by assuming the forms

$$\widetilde{u}_c(s,y,p) = A_n(s,p)e^{-n(s,p)y}$$

$$\widetilde{v}_c(s,y,p) = B_n(s,y,p)e^{-n(s,p)y}$$

$$(7)$$

Substitution of (7) in (5) defines the four characteristic roots

$$n(s,p) = \pm\alpha, \quad \pm\beta, \quad \text{where}$$

$$\alpha = \sqrt{k_d^2 - s^2} \quad \text{and} \quad \beta = \sqrt{k_s^2 - s^2}$$

$$(7a)$$

The particular integrals for (5) can be found in a variety of ways. Convenient here was the use of a Laplace transform on y, along with a convolution, which gave

$$\widetilde{u}_p(s,y,p) = \frac{1}{k_s^2}\int_0^y\left\{\left[\frac{s^2}{\alpha}\sinh\alpha(y-y') + \beta\sinh\beta(y-y')\right]f(y')\right.$$

$$\left. - k^2 s[\cosh\alpha(y-y') - \cosh\beta(y-y')]g(y')\right\}dy'$$

$$\widetilde{v}_p(s,y,p) = \frac{1}{k_d^2}\int_0^y\left\{\left[\alpha\sinh\alpha(y-y') + \frac{s^2}{\beta}\sinh\beta(y-y')\right]g(y)\right.$$

$$\left. + \frac{s}{k^2}[\cosh\alpha(y-y') - \cosh\beta(y-y')]f(y')\right\}dy'$$

$$(8)$$

It follows from (7), (7a), and (8) that the general transformed solutions for (5) are

$$\tilde{u}(s, y, p) = A_{\alpha}(s, p)e^{-\alpha y} + A_{-\alpha}(s, p)e^{\alpha y} + A_{\beta}(s, p)e^{-\beta y}$$

$$+ A_{-\beta}(s, p)e^{\beta y} + \tilde{u}_p(s, y, p)$$

$$\tilde{v}(s, y, p) = \varphi_{\alpha}\left[A_{\alpha}(s, p)e^{-\alpha y} - A_{-\alpha}(s, y)e^{\alpha y} \right] + \varphi_{\beta}\left[A_{\beta}(s, p)e^{-\beta y} \right.$$

$$\left. - A_{-\beta}(s, p)e^{\beta y} \right] + \tilde{v}_p(s, y, p)$$

$$\tag{9}$$

where

$$\varphi_{\alpha} = \frac{B_{\alpha}(s, p)}{A_{\alpha}(s, p)} = -\frac{\alpha}{s}, \qquad \varphi_{\beta} = \frac{B_{\beta}(s, p)}{A_{\beta}(s, p)} = \frac{s}{\beta} \tag{9a}$$

stems from the fact that the two algebraic equations yielding (7a) must hold for all values of n.

Although certainly not a restriction to, particular interest here will be in symmetric excitation of the plate (w. r. t. the mid-plane, y=0). On this basis it is only necessary to consider half of the plate, i. e. $0 \leq y \leq h$. Narrowing further to plate face-type loadings, as a special case, the boundary conditions at y=h are, from the second and third equations of (2) and their transformation in (6),

$$\frac{\tilde{\sigma}_y(s, h, p)}{\lambda + 2\mu} = \left(\frac{k^2 - 2}{k^2}\right)[s\tilde{u}(s, h, p) - \bar{u}(0, h, p)] + \tilde{v}_y(s, h, p) = \frac{\sigma_0 \tilde{F}(s)\bar{G}(p)}{\lambda + 2\mu} = T(s, p)$$

$$\frac{\tilde{\sigma}_{xy}(s, h, p)}{\mu} = \tilde{u}_y(s, h, p) + s\tilde{v}(s, h, p) - \bar{v}(0, h, p) = 0 \tag{10}$$

where $\sigma_0 F(x)G(x)$ is the symmetric (w. r. t. the plate mid-plane) normal stress input on the plate face $y = h$, $F(x)$ and $G(t)$ being arbitrary, but transformable, functions, and σ_0 an inherently negative magnitude constant of dimensions force/unit length. At y=0 the conditions are

$$\frac{\widetilde{\overline{\sigma}}_{xy}(s, 0, p)}{\mu} = \widetilde{\overline{u}}_y(s, 0, p) + s\widetilde{\overline{v}}(s, 0, p) - \overline{v}(0, 0, p) = 0$$

$$\widetilde{\overline{v}}(s, 0, p) = 0$$

(11)

Substitution of (9) in (10) and (11), determines the coefficients $A_{\pm\alpha, \pm\beta}$.
They are

$$A_\alpha = A_{-\alpha} = A(s, p) = \frac{A_N(s, p)}{R(s, p)}$$

$$A_\beta = A_{-\beta} = B(s, p) = \frac{B_N(s, p)}{R(s, p)}$$

(12)

where

$$2A_N(s, p) = -s\left[k^2(2s^2 - k_s^2)\sinh\beta h \cdot I + 2s\beta\cosh\beta h \cdot J\right]$$

$$2B_N(s, p) = -\beta\left[2k^2 s\alpha\sinh\alpha h \cdot I - (2s^2 - k_s^2)\cosh\alpha h \cdot J\right]$$

$$R(s, p) = \left(2s^2 - k_s^2\right)^2 \cosh\alpha h \sinh\beta h + 4s^2\alpha\beta\sinh\alpha h \cosh\beta h$$

and

$$I = T(s, p) + \frac{1}{k_s^2}\int_0^h \left\{\frac{s}{k^2}\left[\frac{(2s^2 - k_s^2)}{\alpha}\sinh\alpha(h - y') + 2\beta\sinh\beta(h - y')\right]f(y')\right.$$

$$+ \left[(2s^2 - k_s^2)\cosh\alpha(h - y') - 2s^2\cosh\beta(h - y')\right]g(y')\Big\}dy'$$

$$+ \left(\frac{k^2 - 2}{k^2}\right)\overline{u}(0, h, p)$$

$$J = \frac{1}{k_s^2}\int_0^h \left\{\left[2s^2\cosh\alpha(h - y') - (2s^2 - k_s^2)\cosh\beta(h - y')\right]f(y')\right.$$

$$+ \frac{k^2 s}{\beta}\left[2\alpha\beta\sinh\alpha(h - y') + (2s^2 - k_s^2)\sinh\beta(h - y')\right]g(y')\Big\}dy' - \overline{v}(0, h, p)$$

where it should be pointed out that A_N and B_N contain the edge unknowns, through $f(y)$ and $g(y)$, and the corner displacements $\overline{u}(0, h, p)$ and $\overline{v}(0, h, p)$, in I and J. Formally, however, the transformed solution on t can be written as

$$\overline{u}(x, y, p) = \frac{1}{2\pi i} \int_{Br_s} \left[\widetilde{\overline{u}}_c(s, y, p) + \widetilde{\overline{u}}_p(s, y, p) \right] e^{sx} ds$$

and

$$(13)$$

$$\overline{v}(x, y, p) = \frac{1}{2\pi i} \int_{Br_s} \left[\widetilde{\overline{v}}_c(s, y, p) + \widetilde{\overline{v}}_p(s, y, p) \right] e^{sx} ds$$

where the complementary functions are given by

$$\widetilde{\overline{u}}_c(s, y, p) = 2 \left[A(s, p) \cosh \alpha y + B(s, p) \cosh \beta y \right]$$

and

$$\widetilde{\overline{v}}_c(s, y, p) = 2 \left[\frac{\alpha}{s} A(s, p) \sinh \alpha y - \frac{s}{\beta} B(s, p) \sinh \beta y \right]$$

where Br_s is the Bromwich contour in the right half s-plane.

Exact Inversion Procedure for Steady Propagation Case

Since the integrands in (13) are even functions of both α and β, they are singled valued functions of s everywhere in the s-plane, for an arbitrary, but fixed, value of p. This reflects the finiteness of the domain in y. Setting $p=i\omega$, (13) represents the formal solution for the steady wave propagation case in the present problem. Further if $s=i\varkappa$, R becomes, in this case,

$$R(s, p) \Big|_{\substack{p=i\omega \\ s=i\varkappa}} = i \left[(\varkappa^2 - \beta'^2)^2 \cos \alpha' h \sin \beta' h + 4\varkappa^2 \alpha' \beta' \sin \alpha' h \cos \beta' h \right] \quad (14)$$

where

$$\alpha' = \sqrt{\frac{\omega^2}{c_d^2} - \varkappa^2}, \quad \text{and} \quad \beta' = \sqrt{\frac{\omega^2}{c_s^2} - \varkappa^2},$$

ω and $\varkappa$ being frequency and wave number, respectively. The bracket on the right hand side, set equal to zero, may be recognized as the classical Rayleigh-Lamb frequency equation for symmetric straight-crested harmonic waves in an infinite elastic plate, where α' and β' are the thickness wave numbers in the plate for dilatational and equivoluminal waves, respectively.

Clearly, then, the roots $s_k(\omega) = i\varkappa_k(\omega)$ of R in (14), which are known to be simple zeros, infinite in number, are the modes, or branches, of wave propagation in the present problem, at least for the steady case. Through the efforts of Holden, Mindlin, Onoe, and others, the nature of these modes are now well-known. A recent study by Onoe, McNiven, and Mindlin [5] on the closely related Pochhammer frequency equation[3] for axially symmetric waves in an infinite circular cylindrical rod gives a detailed account of the main features of these spectra. As Fig. 6 of [5] (Fig. 2 here) exhibits, it is known that for real frequency (dimensionless Ω in Fig. 2) there are an infinite number of branches (or roots), each of which is a continuous one to one relation between frequency and wave number [dimensionless $\zeta = \xi + i\eta$ in Fig. 2, which is essentially $\varkappa h$ in (14)] over the frequency domain from ∞ to $-\infty$. Over this domain the wave number may be real or take on imaginary and complex values, as Fig. 2 exhibits. It is easily shown that $\zeta(\Omega)$ and $-\overline{\zeta}(\Omega)$ are conjugate roots of (14), the latter being a reflection in the plane $\xi = 0$. These roots are indicated in the Figure. Now, since ζ occurs only as ζ^2 in (14), $-\zeta(\Omega)$ and $\overline{\zeta}(\Omega)$ are also roots, these being

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[3]Equations (1a) and (1b) in [6] exhibit the closeness in form of the two equations.

reflections of the former two in the plane $\eta=0$. So for $\Omega \geq 0$, a complex segment of a branch has images in both the planes $\xi=0$ and $\eta=0$, an imaginary segment an image in the plane $\eta=0$, and a real segment an image in the plane $\xi=0$. Fig. 2 offers a scheme for following a particular branch (numbered) continuously through its various segments as Ω progresses from ∞ to zero or vice versa[4]. It follows from the foregoing discussion that once the edge unknowns are determined, inversion of the complementary functions in (13) can be accomplished with simple residue theory. Assuming then, the inverse of $\tilde{u}_p$ and $\tilde{v}_p$ can be found, this would complete the solution for the case of steady propagation in the present problem.

Exact Inversion Procedure for Transient Case.

Extension to the transient solution requires inverting the integrals in

$$\left. \begin{array}{l} u(x, y, t) = \dfrac{1}{2\pi i} \displaystyle\int_{Br_p} \overline{u}(x, y, p)e^{pt}dp \\[2em] v(x, y, t) = \dfrac{1}{2\pi i} \displaystyle\int_{Br_p} \overline{v}(x, y, p)e^{pt}dp \end{array} \right\} \tag{15}$$

where $\overline{u}$ and $\overline{v}$ are given by (13), Br_p being the Bromwich contour in the right half p-plane. If now it is assumed the roots $s_k(\omega)=i\varkappa_k(\omega)$ are analytically continuable from μ on the imaginary axis, i.e. roots of the right hand side of (14), to the corresponding set $s_k(p)=i\varkappa_k(p)$ on the Bromwich contour Br_p, (15) can be inverted by termwise contour integration of the terms in the series resulting from the inversion of (13). This method of inverting the

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[4] Since Ω occurs as Ω^2 in (14), $-\Omega$ will also yield the spectra shown in Fig. 2, i. e. the reflection in the plane $\Omega=0$. Such modes usually play a part in deriving the solution for a problem (like the present one). The reflection principle, however, reduces the solution to dependence on $\Omega \geq 0$ only.

double integral transform, i. e. inversion of spatial transform first, was discussed further and used by Lloyd and Miklowitz [7] in related work. It is based also on the assumption that there are no singularities in $\bar{u}$ and $\bar{v}$ in the right half p-plane, $\mathrm{Re}\, p > 0$. Clearly such singularities would be inconsistent with the fact that bounded long time, or static solutions, for the present problems do exist. It follows that the contour integration can be carried out along the imaginary axis in the p-plane, yielding a transient solution in the form of a series of integrals, each term of which represents a branch of the frequency equation depicted in Fig. 2.

Boundedness Condition; Integral Equations for Edge Unknowns.

It follows from the discussion of the roots $x_k(\omega)$ of R in (14) and their continuation to $x_k(p)$, and Fig. 2 depicting them, that there are two sets of complex branches, i. e. $s_j(p) = ix_j(p)$, and their conjugates $s_j(p) = \overline{ix}_j(p)$, that satisfy

$$\mathrm{Re}\, s_k(p) > 0, \quad (k = j) \tag{16}$$

which in terms of Fig. 2 is

$$\mathrm{Re}\left[i(\xi + i\eta)\right] > 0$$

or

$$\eta = \mathrm{Im}\, \zeta < 0, \tag{16a}$$

i. e. reflections of the complex branches in Fig. 2 in the plane $\eta=0$. It is clear that these roots of R, which lie in the right half s-plane, would give, through the residue evaluation of (13), exponentially unbounded waves at $x \to \infty$, hence violating the condition (4). It follows that these unbounded waves can be eliminated by requiring the residues in (13), associated with them, be zero, i. e.

$$A_N(s,p)\Big|_{s=s_j(p)} = B_N(s,p)\Big|_{s=s_j(p)} = 0 \tag{17}$$

where $s_j(p)$ are the two sets of roots of $R(s,p)$ in (12) satisfying (16).

Substitution of A_N and B_N from (12) into (17) gives two equations for I and J, for each $s_j(p)$. Compatibility of these equations requires that

$$J_j - P_j I_j = 0 \tag{18}$$

where J_j and I_j are J and I in (12) for $s=s_j(p)$, and

$$P_j = \frac{2k^2 s_j \alpha_j \sinh\alpha_j h}{\left(2s_j^2 - k_s^2\right)\cosh\alpha_j h} = -\frac{k^2\left(2s_j^2 - k_s^2\right)\sinh\beta_j h}{2s_j\beta_j \cosh\beta_j h}$$

$$\alpha_j = \sqrt{k_d^2 - s_j^2} \,, \qquad \beta_j = \sqrt{k_s^2 - s_j^2}$$

Now arguing on the basis of Lerch's theorem[5], that for the present p can be assumed to be real, and noting again the conjugate nature of the roots $s_j(p)$ satisfying (16), the boundedness condition (18) can be expanded into

$$\begin{Bmatrix} Re \\ Im \end{Bmatrix} \left\{ \frac{1}{k_s^2} \int_0^h \left[\left[2s_j^2 \cosh\alpha_j(h-y') - \frac{s_j\left(2s_j^2 - k_s^2\right)}{k^2\alpha_j} P_j \sinh\alpha_j(h-y') \right.\right.\right.$$

$$\left. - \left(2s_j^2 - k_s^2\right)\cosh\beta_j(h-y') - \frac{2s_j\beta_j}{k^2} P_j \sinh\beta_j(h-y') \right] f(y')$$

$$+ \left[2k^2 s_j\alpha_j \sinh\alpha_j(h-y') - \left(2s_j^2 - k_s^2\right)P_j \cosh\alpha_j(h-y') \right.$$

$$\left.\left. + \frac{k^2 s_j\left(2s_j^2 - k_s^2\right)}{\beta_j}\sinh\beta_j(h-y') + 2s_j^2 P_j \cosh\beta_j(h-y') \right] g(y') \right\} dy'$$

$$\left. - \bar{v}(0,h,p) - P_j\left[T(s_j,p) + \left(\frac{k^2-2}{k^2}\right)\bar{u}(0,h,p)\right] \right\} = 0 \tag{19}$$

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[5] An inverse Laplace transform is unique provided the transform is known for all real values of p for $p > p_0$ (see Widder [8]).

(19) are two coupled integral equations for the edge unknowns for each set of parameters p and $s_j(p)$. For a particular problem the edge conditions would reduce the number of edge unknowns in (19), and assuming a solution of these equations can be found for the remaining edge unknowns, this would complete the formal solutions given by (13) and (15) as will soon be demonstrated.

§3. PROBLEM OF PLATE WITH BUILT-IN EDGE

As a first example the case of a plate with a built-in edge was treated. As illustrated in Fig. 3 the plate is excited symmetrically by two suddenly applied normal line loads on its faces, a distance a from the edge. This particular loading is expressed in the first of (10) through

$$\frac{\sigma_y(x, h, t)}{\lambda+2\mu} = T(x, t) = \frac{\sigma_0 \delta(x-a)G(t)}{\lambda+2\mu} \tag{20a}$$

where $\delta(x-a)$ is the symmetrical delta function, so that $T(s, p)$ in (10), and later equations, is

$$T(s, p) = \frac{\sigma_0 \overline{G}(p) e^{-sa}}{(\lambda+2\mu)} \tag{20b}$$

The edge conditions are given by

$$\left.\begin{array}{l} u(0, y, t) = v(0, y, t) = 0 \\ u_y(0, y, t) = v_y(0, y, t) = 0 \end{array}\right\} \tag{21a}$$

so that

$$\left.\begin{array}{l} \overline{u}(0, y, p) = \overline{v}(0, y, p) = 0 \\ \overline{u}_y(0, y, p) = \overline{v}_y(0, y, p) = 0 \end{array}\right\} \tag{21b}$$

(21), it may be noted, reduces the unknowns in the two equations (19) to two, $\overline{u}_x(0, y, p)$ and $\overline{v}_x(0, y, p)$, hence, apparently, a well-posed and simpler

problem in the present analysis than the free edge (or edge loaded) prob-

lem. In the latter one would have to deal with the four unknowns $\overline{u}(0, y, p)$,

$\overline{u}_x(0, y, p)$, $\overline{v}(0, y, p)$, and $\overline{v}_x(0, y, p)$.

Determination of Edge Unkowns for Long-Time Solution.

During the course of this work the author learned of a paper by

Benthem [9][6] who exploited a similar boundedness condition, to that used

here, to solve related static problems in strips governed by the biharmonic

equation on the Airy stress function for plane stress[7]. Of particular interest

in Benthem's technique is his use of eigenfunction expansions (Fourier series

in his problems), and singular terms (where needed), to represent the edge

unknowns. This scheme reduces the boundedness condition (in his work) to

a finite set of algebraic equations for the coefficients of the Fourier series

and the singular terms. With numerical evolutions, Benthem shows that

only a very limited number of the coefficients need be determined. Benthem's

comparisons of his results with those in the early work of Knein [11] for

the problem of the strip, or plate, with a built-in edge, which the latter

obtained by other means, show very good agreement. It was therefore of

interest to see if Benthem's technique could be extended to finding solutions

of (19), and in particular, for the problem of the built-in edge being treated

here.

Since one would expect the static solution as the long-time limit in

the present problem, it seemed reasonable to formulate representations for

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[6] The author is indebted to his colleague Professor James Knowles for
pointing out Benthem's work.

[7] Benthem points out that Doetsch [10], in a strip problem governed by
Laplace's equation, employed a boundedness condition in a manner
related to that in Benthem's, and hence the present work.

the edge unknowns, with expansions similar in form to those used by Benthem for his problem involving a built-in edge, and derive the long-time solution. On this basis, it was assumed that

$$f(y) = k^2 \bar{u}_x(0, y, p) = -k^2 \left[\frac{a_0(p)}{h\left(1 - \frac{y}{h}\right)^q} + \sum_{n=1,3,5,\ldots} a_n(p) \cos\frac{n\pi y}{2h} \right] \tag{22a}$$

$$g(y) = \left(\frac{1}{k^2}\right)\bar{v}_x(0, y, p) = \frac{Ca_0(p)}{k^2 h} \left[\frac{1}{\left(1 - \frac{y}{h}\right)^q} - \left(1 - \frac{y}{h}\right) \right] + \sum_{n=2,4,6,\ldots} \frac{b_n(p)}{k^2} \sin\frac{n\pi y}{2h} \tag{22b}$$

These are similar in form to Benthem's equations (6.1) for the edge normal and shear stress. In comparing the two it may be noted whereas he had constant coefficients, here a_0, a_n, and b_n are functions of the parameter p. The power q of the corner singularities in (22) is taken to be that found in the eigenfunction problem for the clamped-free wedge. This was treated by Knein [11] for the right angle wedge or corner (case in [9] and here), and by Williams [12] for more general angle corners. As these works show q depends only on Poisson's ratio for a fixed corner angle. Hence, although Knein treated a case of the built-in edge (and plane strain) for a uniform edge compression, whereas here lateral loads are involved, a particular q holds for both problems[8]. Knein gives curves for q, and $\left|\frac{\sigma_{xy}}{\sigma_x}\right|$ along x=0 (near the corner) which determines C in (22b), as a function of Poisson's ratio ν. Following Knein, ν is taken to be 0.2433 here, which makes q very close to $1/4$, and C = 0.740.

   Then by substituting (22) in (19), and with the aid of the integrations

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[8] The same q is also applicable in Benthem's problem, which involves an applied uniform tension load over y at $x \to \infty$. Since Benthem's case is one of plane stress, Poisson's ratio must be different.

$$\int_0^h \begin{Bmatrix} \cosh \\ \sinh \end{Bmatrix} \{\alpha_j(h-y')\} \cos\frac{n\pi y'}{2h}\, dy' = \frac{1}{A_j^n}\left\{ \begin{matrix} (-)^{\frac{n-1}{2}}\frac{n\pi}{2h}+\alpha_j\sinh\alpha_j h \\[2mm] \alpha_j\cosh\alpha_j h \end{matrix} \right\}, \quad n=1,3,5,.. \quad (23a)$$

and

$$\int_0^h \begin{Bmatrix} \sinh \\ \cosh \end{Bmatrix} \{\alpha_j(h-y')\} \sin\frac{n\pi y'}{2h}\, dy' = \frac{1}{A_j^n}\left\{ \begin{matrix} \frac{n\pi}{2h}\sinh\alpha_j h \\[2mm] \frac{n\pi}{2h}[-(-)^{n/2}+\cosh\alpha_j h] \end{matrix} \right\}, \quad n=2,4,6,.. \quad (23b)$$

and

$$\int_0^h (1-y'/h)\begin{Bmatrix} \cosh \\ \sinh \end{Bmatrix} \{\alpha_j(h-y')\}\, dy' = \frac{1}{\alpha_j^2 h}\left\{ \begin{matrix} \alpha_j h\sinh\alpha_j h-\cosh\alpha_j h+1 \\[2mm] \alpha_j h\cosh\alpha_j h-\sinh\alpha_j h \end{matrix} \right\} \quad (24a)$$

and

$$\int_0^h \begin{Bmatrix} \sinh \\ \cosh \end{Bmatrix} \{\alpha_j(h-y')\}\frac{dy'}{h(1-y'/h)^{1/4}} = \int_0^1 \begin{Bmatrix} \sinh\alpha_j h\zeta \\ \cosh\alpha_j h\zeta \end{Bmatrix}\frac{d\zeta}{\zeta^{1/4}} = \begin{Bmatrix} S_{\alpha_j} \\ C_{\alpha_j} \end{Bmatrix} \quad (24b)$$

where $A_j^n = \alpha_j^2 + n^2\pi^2/4h^2$, and the aid of the duplicate set of integrations to (23) and (24), where β_j and $B_j^n = \beta_j^2 + n^2\pi^2/4h^2$ replace α_j and A_j^n, respectively, (19) can be reduced to the algebraic equations

$$\begin{Bmatrix} \text{Re} \\ \text{Im} \end{Bmatrix} \left\{ M_j^0(s_j,p)a_0(p) - \sum_{n=1,3,5,..} (-)^{\frac{n-1}{2}} M_j^n(s_j,p)a_n(p) \right.$$

$$\left. + \sum_{n=2,4,6,..} (-)^{\frac{n}{2}} N_j^n(s_j,p)b_n(p) - Q_j(s_j,p) \right\} = 0 \quad (25)$$

where

$$M_j^0(s_j, p) = \left(2s_j^2 - k_s^2\right)\left(C_{\beta_j} - \frac{\sinh\beta_j h}{\cosh\beta_j h}S_{\beta_j}\right) - 2s_j^2\left(C_{\alpha_j} - \frac{\sinh\alpha_j h}{\cosh\alpha_j h}S_{\alpha_j}\right)$$

$$+ \frac{Cs_j}{k^2}\left[2\alpha_j\left(S_{\alpha_j} - \frac{\sinh\alpha_j h}{\cosh\alpha_j h}C_{\alpha_j}\right) + \frac{\left(2s_j^2 - k_s^2\right)}{\beta_j}\left(S_{\beta_j} - \frac{\sinh\beta_j h}{\cosh\beta_j h}C_{\beta_j}\right)\right.$$

$$\left.- \frac{1}{h}\left\{\frac{2}{\cosh\alpha_j h} + \frac{\left(2s_j^2 - k_s^2\right)}{\beta_j^2\cosh\beta_j h} - \frac{2\sinh\alpha_j h}{\alpha_j h\cosh\alpha_j h} - \frac{\left(2s_j^2 - k_s^2\right)}{\beta_j^3 h}\frac{\sinh\beta_j h}{\cosh\beta_j h}\right\}\right]$$

$$M_j^n(s_j, p) = \frac{n\pi}{2h}\left[\frac{2s_j^2}{A_j^n} - \frac{\left(2s_j^2 - k_s^2\right)}{B_j^n}\right]$$

$$N_j^n(s_j, p) = \frac{n\pi s_j}{2hk^2}\left[\frac{2\alpha_j\sinh\alpha_j h}{A_j^n\cosh\alpha_j h} + \frac{\left(2s_j^2 - k_s^2\right)\sinh\beta_j h}{\beta_j B_j^n\cosh\beta_j h}\right]$$

$$Q_j(s_j, p) = \frac{2\sigma_0 k_d^2 s_j\alpha_j\sinh\alpha_j h}{\mu\left(2s_j^2 - k_s^2\right)\cosh\alpha_j h} \cdot \overline{G}(p)e^{-s_j a} \quad .$$

(25) can be solved for the unknowns $a_0(p)$, $a_n(p)$, and $b_n(p)$. That is, for a certain number of unknowns, $a_0(p)$, $a_1(p)$, $a_3(p), \cdots$, $b_2(p)$, $b_4(p), \cdots$, a matching number of $s_j(p)$ (which are infinite in number; see Fig. 2) are available to give a sufficient number of equations from (25) to solve for these unknowns. Benthem shows in the static case by progressively taking more and more unknowns (constants like a_n's and b_n's here), he gets convergence, numerically, to an a_N and b_N value, where N is a relatively small number. As he points out this would have to be the case since the series representations in (22), obviously, are not being called upon to represent the singular terms there, i. e. if they were, then one would expect the convergence to be quite slow. It follows that in the long-time solution for the present problem analogous behavior could be expected.

The long-time solution can be derived from (15) by making use of the underlying Tauberian theorem, i. e.

$$\lim_{t \to \infty} \begin{Bmatrix} u(v, y, t) \\ v(x, y, t) \end{Bmatrix} = \lim_{p \to 0} \left\{ \frac{1}{2\pi i} \int_{Br_p} \begin{bmatrix} \overline{u}(x, y, p) \\ \overline{v}(x, y, p) \end{bmatrix} e^{pt} dp \right\} \tag{26}$$

Hence, the solution of (25) must be obtained for $p \to 0$ in order to determine $\overline{u}$ and $\overline{v}$ in (26). The roots $s_j(p)$ of $R(s, p)$ in (12) can be found by evaluating

$$\lim_{p \to 0} \left\{ \frac{R}{k_s^2} \right\} = \lim_{p \to 0} \left\{ c_s^2 \frac{\partial R}{\partial p^2} \right\} \tag{27}$$

which gives

$$\lim_{p \to 0} \left\{ \frac{R}{k_s^2} \right\} = - \frac{is^2}{2(1-\nu)} (\sin 2sh + 2sh) \tag{28}$$

Hence the $s_j(p)$ are selected from the zeros of

$$r(s) = \sin z + z, \quad z = 2sh = x' + iy' \tag{29}$$

satisfying (16), i. e. $\mathrm{Re}\, s_j > 0$. The zeros of $r(s)$ are an ordered infinite set, corresponding to the piercing points of R in the plane $\Omega = 0$ as Fig. 2 indicates. $r(s)$ is a well known function in the analyses of elastostatic plate problems. Hence its occurrence here, and in Benthem's work is not surprising. Robbins and Smith [13] give the first ten roots to 6 decimals, and the asymptotic behavior of large roots. The behavior of the slopes of $s_j(p)$ can be found by noting that along these branches

$$dR = \frac{\partial R}{\partial p} dp + \frac{\partial R}{\partial s} ds = 0$$

and therefore

$$\lim_{p \to 0} \left\{ \frac{1}{p} \frac{ds}{dp} \right\} = \lim_{p \to 0} \left\{ - \frac{\frac{\partial R}{\partial p}}{p \frac{\partial R}{\partial s}} \right\} \tag{30}$$

which from (28) is

$$\lim_{p \to 0} \left\{ \frac{1}{p} \frac{ds}{dp} \right\} = - \frac{2r(s)}{p^2 r'(s)} = 0 \tag{31}$$

since $r(s) = 0$, and $r'(s) \neq 0$, the prime denoting differentiation w. r. t. s.
It follows that ds/dp in (31) must vanish in the limit, i. e. these branches
are normal to the plane $\Omega = 0$. It follows that the zeros of $r(s)$ are a good
approximation to the $s_j(p)$ for a range of p, small but > 0. This is impor-
tant to the validity of (26), hence the long-time solution being sought. It
should also be pointed out that (28) directly proves the existence of the
zeros of $R(s, p)$, at least over the domain of small complex p, that was
argued earlier on the basis of the analytical continuation of these zeros,
generally, from the domain of imaginary p [Re discussion after (15)].

With the $s_j(p)$ determined from the zeros of $r(s)$ it remains to
approximate the coefficients in (25) for p small, and then solve for the
unknowns. Expanding α_j, β_j, and quantities dependent on these, for small
p, there results

$$\alpha_j(s_j, p) = is_j \left[1 - \frac{k_d^2}{2s_j^2} + O(p^4) \right], \quad \beta_j(s_j, p) = is_j \left[1 - \frac{k_s^2}{2s_j^2} + O(p^4) \right]$$

$$\sinh \begin{Bmatrix} \alpha_j h \\ \beta_j h \end{Bmatrix} = \sinh is_j h - \frac{ih}{2s_j} \begin{Bmatrix} k_d^2 \\ k_s^2 \end{Bmatrix} \cosh is_j h + O(p^4)$$

$$\cosh \begin{Bmatrix} \alpha_j h \\ \beta_j h \end{Bmatrix} = \cosh is_j h - \frac{ih}{2s_j} \begin{Bmatrix} k_d^2 \\ k_s^2 \end{Bmatrix} \sinh is_j h + O(p^4)$$

which can be used to obtain

$$\begin{Bmatrix} S_{\alpha_j} \\ S_{\beta_j} \end{Bmatrix} = S_{is_j} - \frac{ih}{2s_j} \begin{Bmatrix} k_d^2 \\ k_s^2 \end{Bmatrix} \gamma_{is_j} + O(p^4)$$

$$\begin{Bmatrix} C_{\alpha_j} \\ C_{\beta_j} \end{Bmatrix} = C_{is_j} - \frac{ih}{2s_j} \begin{Bmatrix} k_d^2 \\ k_s^2 \end{Bmatrix} \delta_{is_j} + O(p^4)$$

where S_{is_j} and C_{is_j} are given by the second integral in (24b), where $\alpha_j = \beta_j = is_j$, and

$$\gamma_{is_j} = \int_0^1 \zeta^{3/4} \cosh is_j h\zeta \, d\zeta$$

$$\delta_{is_j} = \int_0^1 \zeta^{3/4} \sinh is_j h\zeta \, d\zeta$$

These expansions lead to

$$\left.\begin{aligned}
\lim_{p \to 0} M_j^0(s_j, p) = &\left(k_d^2 - k_s^2\right) \left\{ is_j h \left[\delta_{is_j} - \varkappa_{is_j} \left(\gamma_{is_j} + i \frac{S_{is_j}}{s_j h} \right) \right] \right. \\
&+ \frac{C}{k^2} \left\{ s_j h \left[\gamma_{is_j} - \varkappa_{is_j} \left(\delta_{is_j} + i \frac{C_{is_j}}{s_j h} \right) \right] - \frac{i\varkappa_{is_j}}{\cosh is_j h} \right\} \Bigg\} \\
&- k_d^2 \left\{ k^2 \left(C_{is_j} - \varkappa_{is_j} S_{is_j} \right) + \frac{C}{k^2} \left\{ i \left(S_{is_j} - \varkappa_{is_j} C_{is_j} \right) - \frac{k^2}{s_j h \cosh is_j h} \right\} \right\} \\
&+ \frac{iC \left(3k_s^2 - 2k_d^2 \right) \varkappa_{is_j}}{k^2 s_j^2 h^2} + O(p^4)
\end{aligned}\right\} \quad (32)$$

$$\lim_{p \to 0} M_j^n(s_j, p) = \frac{n\pi}{2h\,U_j^n} \left[\frac{2s_j^2}{U_j^n} \left(k_s^2 - k_d^2 \right) + k_s^2 \right] + O(p^4)$$

$$\lim_{p \to 0} N_j^n(s_j, p) = \frac{in\pi \varkappa_{is_j}}{2hk^2 U_j^n} \left[\frac{2s_j^2}{U_j^n} \left(k_s^2 - k_d^2 \right) - k_s^2 \right] + O(p^4) \qquad (32)$$

and

$$\lim_{p \to 0} Q_j(s_j, p) = i(\sigma_0/\mu) e^{-s_j a} \varkappa_{is_j} \overline{G}(p) k_d^2 \left[1 + O(p^2) \right]$$

where

$$\varkappa_{is_j} = \frac{\sinh is_j h}{\cosh is_j h}; \quad \text{and} \quad U_j^n = \frac{n^2 \pi^2}{4h^2} - s_j^2 .$$

Since the real and imaginary parts of the leading terms in (32) behave as

$$\lim_{p \to 0} \begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix} \left\{ M_j^0, M_j^n, N_j^n \right\} = O(p^2)$$

and

$$\lim_{p \to 0} \begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix} Q_j = O\left[p^2 \overline{G}(p) \right] ,$$

their substitution in (25), shows that a solution of the latter, say by Cramer's rule, will have the behavior

$$\lim_{p \to 0} \left[a_0(p), a_1(p), a_3(p), \dots b_2(p), b_4(p), b_6(p), \dots \right] \sim \overline{G}(p) \qquad (33)$$

(33) is an important result since it shows the long-time behavior of the unknown strains at the edge (x=0), $u_x(0, y, t)$ and $v_x(0, y, t)$, according to (22), is the same as the input loads on the plate faces. Further if G(t) is

set equal to the step H(t), it is clear these edge unknowns have the same form as in Benthem's work, i.e. the unknowns become constants. (33), and (32), show the numerical problem remaining to completely define the unknowns is simple. One would evaluate forms such as

$$\frac{a_3(p)}{\overline{G}(p)} = \frac{\left| X_j^n \right|}{\left| Y_j^n \right|} \tag{34}$$

where the R. H. S. depends only on n and s_j, and not on p [s_j are the roots of r(s) in (29)], and use Benthem's numerical convergence procedure for the unknowns, i.e. repetition of the solution, with a larger number of unknowns, until convergence to the final value of each unknown is reached. Benthem's example shows convergence is reached quickly. Except for carrying out this numerical work, the analysis given here makes it possible to define the formal long-time solution (26). In addition to the very long-time (static) solution, (26) will yield the low frequency-long wave dynamic contribution from the lowest real mode (see Fig. 2) that occurs earlier. Since p is also small for this contribution, it can be derived from (26) because it is compatible with, and hence can be based on, the approximations contained in (32) and (33). The numerical work required to define the unknowns, and the study of the low-frequency and static contributions to the long-time solution (26), are planned, and will be reported in a sequel to this work.

Formal Long-Time Solution.

With the transformed edge unknowns $\overline{u}_x(0, y, p)$ and $\overline{v}_x(0, y, p)$ determined for small p, through the analysis given in the previous section, the formal solution for long-time given by (26) can be determined, i.e.

$\overline{u}(x, y, p)$ and $\overline{v}(x, y, p)$ in (26) can be determined. Returning to (13), the integrands there, corresponding to (22), can be obtained by substituting the latter into (8) and (12) and performing the integrations indicated. The resulting double integral transforms for $\widetilde{u}_c$ and $\widetilde{v}_c$ are given by

$$\widetilde{u}_c(s, y, p) = \sum_{i=1}^{4} \widetilde{u}_{ci}(s, y, p) = \sum_{i=1}^{4} \left[\widetilde{u}_{ci}^R + \widetilde{u}_{ci}^0\right]$$

$$\widetilde{v}_c(s, y, p) = \sum_{i=1}^{4} \widetilde{v}_{ci}(s, y, p) \tag{35}$$

where

$$\widetilde{u}_{c1}^R = \frac{1}{R}\left[I_u I_1 + J_u J_1\right]$$

$$\widetilde{u}_{c1}^0 = \left[\frac{s^2}{k_d^2} \sum_{n=1,3,5..} \frac{a_n(p)}{\alpha_n} - \frac{s}{k_s^2} \sum_{n=2,4,6..} \frac{n\pi}{2h} \frac{b_n(p)}{\alpha_n}\right] \cosh\alpha y$$

$$+ \left[\frac{\beta^2}{k_d^2} \sum_{n=1,3,5..} \frac{a_n(p)}{\beta_n} + \frac{s}{k_s^2} \sum_{n=2,4,6..} \frac{n\pi}{2h} \frac{b_n(p)}{\beta_n}\right] \cosh\beta y$$

$$\widetilde{u}_{c2} = \frac{1}{R}\left[I_u I_2 + J_u J_2\right]$$

$$\widetilde{u}_{c3}^R = \frac{Ca_0(p)s}{k_s^2 R}\left[\frac{\left(2s^2 - k_s^2\right)}{\beta h}\{2s^2\cosh\alpha y - 2\beta^2\cosh\beta y\} + \frac{L}{\alpha h}\cosh\alpha y - \frac{M}{\beta h}\cosh\beta y\right.$$

$$\left. + \left\{\frac{2s^2 - k_s^2}{\alpha^2 h^2} - \frac{2s^2}{\beta^2 h^2}\right\}\left\{\left(2s^2 - k_s^2\right)\sinh\beta h \cosh\alpha y + 2\alpha\beta\sinh\alpha h \cosh\beta y\right\}\right]$$

$$\tilde{u}_{c3}^0 = - \frac{Ca_0(p)s}{k_s^2} \left[\frac{\cosh \alpha y}{\alpha^2 h^2} - \frac{\cosh \beta y}{\beta^2 h^2} \right]$$

$$\tilde{u}_{c4} = - \frac{I_u T(s,p)}{R}$$

and

$$\tilde{v}_{c1} = \frac{1}{R} \left[I_v I_1 + J_v J_1 \right]$$

$$\tilde{v}_{c2} = \frac{1}{R} \left[I_v I_2 + J_v J_2 \right]$$

$$\tilde{v}_{c3} = \frac{Ca_0(p)}{k_s^2 R} \left[\frac{2s^2(2s^2-k_s^2)}{\beta h} \{ \alpha \sinh \alpha y + \beta \sinh \beta y \} + \frac{L}{h} \sinh \alpha y + \frac{s^2 M}{\beta^2 h} \sinh \beta y \right.$$

$$\left. + \left\{ \frac{2s^2-k_s^2}{\alpha h^2} - \frac{2s^2 \alpha}{\beta^2 h^2} \right\} \left\{ \left(2s^2-k_s^2\right) \sinh \beta h \sinh \alpha y - 2s^2 \sinh \alpha h \sinh \beta y \right\} \right]$$

$$\tilde{v}_{c4} = - \frac{I_v T(s,p)}{R}$$

and where

$$I_u = s\left(2s^2-k_s^2\right) \sinh \beta h \cosh \alpha y + 2s\alpha\beta \sinh \alpha h \cosh \beta y$$

$$J_u = 2s^2 \beta \cosh \beta h \cosh \alpha y - \beta\left(2s^2-k_s^2\right) \cosh \alpha h \cosh \beta y$$

$$I_v = \alpha\left(2s^2-k_s^2\right) \sinh \beta h \sinh \alpha y - 2s^2 \alpha \sinh \alpha h \sinh \beta y$$

$$J_v = 2s\alpha\beta \cosh \beta h \sinh \alpha y + s\left(2s^2-k_s^2\right) \cosh \alpha h \sinh \beta y$$

$$J_1 = \frac{1}{k_d^2} \sum_{n=1,3,5,..} (-)^{\frac{n-1}{2}} \frac{n\pi}{2h} \left[\frac{2s^2}{\alpha_n} - \frac{2s^2-k_s^2}{\beta_n} \right] a_n(p)$$

$$I_1 = \frac{1}{k_s^2} \sum_{n=2,4,6,..} (-)^{\frac{n}{2}} \frac{n\pi}{2h} \left[\frac{2s^2 - k_s^2}{\alpha_n} - \frac{2s^2}{\beta_n} \right] b_n(p)$$

$$J_2 = -\frac{1}{k_s^2} \left\{ -k^2 \left[2s^2 C_\alpha - \left(2s^2 - k_s^2\right) C_\beta \right] + \frac{Cs}{\beta} \left[2\alpha\beta S_\alpha + \left(2s^2 - k_s^2\right) S_\beta \right] \right\} a_0(p)$$

$$I_2 = -\frac{1}{k_s^2} \left\{ -\frac{k^2 s}{\alpha} \left[\left(2s^2 - k_s^2\right) S_\alpha + 2\alpha\beta S_\beta \right] + C \left[\left(2s^2 - k_s^2\right) C_\alpha - 2s^2 C_\beta \right] \right\} a_0(p)$$

$$L = \left(2s^2 - k_s^2\right)^2 \sinh \alpha h \sinh \beta h + 4s^2 \alpha\beta \cosh \alpha h \cosh \beta h$$

$$M = 4s^2 \alpha\beta \sinh \alpha h \sinh \beta h + \left(2s^2 - k_s^2\right)^2 \cosh \alpha h \cosh \beta h$$

$$\alpha_n = \alpha^2 + \frac{n^2 \pi^2}{4h^2} \, , \qquad \beta_n = \beta^2 + \frac{n^2 \pi^2}{4h^2} \, ,$$

S_α and C_α being the integrals formed from (24b) by dropping the sub j
(S_β and C_β obtained from these by replacing α with β).

The double integral transforms for $\tilde{u}_p$ and $\tilde{v}_p$ are

$$\left. \begin{array}{c} \tilde{u}_p(s, y, p) = \sum_{i=1}^{3} \tilde{u}_{pi} \\[4mm] \tilde{v}_p(s, y, p) = \sum_{i=1}^{3} \tilde{v}_{pi} \end{array} \right\} \tag{36}$$

where

$$\tilde{u}_{p1} = \frac{1}{k_s^2} \left\{ \sum_{n=1,3,5,..} k^2 \left[\frac{s^2 \left(\cos\frac{n\pi y}{2h} - \cosh\alpha y \right)}{\alpha_n} + \frac{\beta^2 \left(\cos\frac{n\pi y}{2h} - \cosh\beta y \right)}{\beta_n} \right] a_n(p) \right.$$

$$\left. - s \sum_{n=2,4,6,..} \left[\frac{\cos\frac{n\pi y}{2h} - \cosh\alpha y}{\alpha_n} - \frac{\cos\frac{n\pi y}{2h} - \cosh\beta y}{\beta_n} \right] \frac{n\pi}{2h} b_n(p) \right\}$$

$$\tilde{u}_{p2} = \frac{a_0(p)}{k_s^2} \int_0^y \left\{ -\frac{k^2}{\alpha} \left[s^2 \sinh\alpha(y-y') + \alpha\beta \sinh\beta(y-y') \right] \right.$$

$$\left. + Cs\left[\cosh\alpha(y-y') - \cosh\beta(y-y') \right] \right\} \frac{dy'}{h(1-y'/h)^{1/4}}$$

$$\tilde{u}_{p3} = -\frac{Ca_0(p)s}{k_s^2} \left\{ \frac{\sinh\alpha y}{\alpha h} - \frac{\sinh\beta y}{\beta h} - \left[\frac{\cosh\alpha y - 1}{\alpha^2 h^2} - \frac{\cosh\beta y - 1}{\beta^2 h^2} \right] \right\}$$

and

$$\tilde{v}_{p1} = -\frac{1}{k_d^2} \left\{ \frac{1}{k^2} \sum_{n=2,4,6,..} \left[\frac{\alpha^2}{\alpha_n} + \frac{s^2}{\beta_n} \right] b_n(p) \sin\frac{n\pi y}{2h} \right.$$

$$\left. + s \sum_{n=1,3,5,..} \left[\frac{1}{\alpha_n} - \frac{1}{\beta_n} \right] \frac{n\pi}{2h} a_n(p) \sin\frac{n\pi y}{2h} \right\}$$

$$\tilde{v}_{p2} = \frac{a_0(p)}{k_d^2} \int_0^y \left\{ \frac{C}{k^2\beta} \left[\alpha\beta \sinh\alpha(y-y') + s^2 \sinh\beta(y-y') \right] \right.$$

$$\left. - s\left[\cosh\alpha(y-y') - \cosh\beta(y-y') \right] \right\} \frac{dy'}{h(1-y'/h)^{1/4}}$$

$$\tilde{v}_{p3} = -\frac{Ca_0(p)}{k_s^2} \left\{ \frac{\cosh\alpha y - 1}{h} + \frac{s^2(\cosh\beta y - 1)}{\beta^2 h} + \frac{y}{h^2} \left[1 + \frac{s^2}{\beta^2} \right] \right\}$$

where in getting $\tilde{u}_{p1}$ and $\tilde{v}_{p1}$, use has again been made of (23), replacing
h with y and α_j with α, etc. It should be noted that in (35) and (36) the
subscripts 1, and 2, correspond to the trigonometric and singular terms
in (22), respectively, and the subscript 3 to the linear term in (22b).
These subscripts help to identify related terms in (35) and (36), i.e. in
$\tilde{u}_c$ and $\tilde{u}_p$ and $\tilde{v}_c$ and $\tilde{v}_p$. (36) shows that all $\tilde{u}_{pi}$ and $\tilde{v}_{pi}$ are zero at y=0 in
accord with (8) from which they stem. It may be noted that the super zero
terms $\tilde{u}_{c1}^{0}$ and $\tilde{u}_{c3}^{0}$ are equivalent, except for sign, to the $\cosh\alpha y$ and $\cosh\beta y$
terms in $\tilde{u}_{p1}$ and $\tilde{u}_{p3}$, respectively, and hence their sum, in each case,
which pertains to y>0, does not contribute to $\tilde{u}$. Indeed the super zero
means that these terms are in the solution for $\tilde{u}$ only at y=0. In the case
of $\tilde{u}_{c2}$ no super zero term appears. Its integral form precludes separating
such a term. A term of this nature is involved, however[9]. The super R
terms are so labled because of the 1/R in them, and they contribute to the
solution for all y. In the case of $\tilde{v}_{ci}$ and $\tilde{v}_{pi}$ like cancellation is involved
for y>0, but since $\tilde{v}=0$ at y=0, there are no super zero type terms, and except
for $\tilde{v}_{c2}$ and $\tilde{v}_{p2}$, all terms have been reduced (to the forms similar to $\tilde{u}_{ci}$
and $\tilde{u}_{pi}$ for y>0).

Important is the fact that (35) and (36) have resulted only from the
assumed form of the unknowns given by (22), i.e. they are not directly
dependent on the assumption of small p. For use in the long-time solution,
however, they will have to be approximated, for small p, to be consistent
with the approximations given by (32) and (33) for the unknown coefficients
$a_0(p)$, $a_n(p)$, and $b_n(p)$, they contain. The implication here is that (22) may

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[9]Such a term is recovered in the near field solution derived later, and shown
to exist for $\tilde{u}$ only at y=0.

be valid for the propagation of other waves, e.g. harmonic waves for certain single frequencies, calculated from (13). This will be investigated in future work.

Approximation for the Near Field.

The near field solution $(x << a)$ can be written by applying the underlying Tauberian theorem for this case to (13), with (35) and (36), expanding terms in the latter for $|sh| >> 1$, with $|k_s^2/s^2| < 1$, making

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = -is \left[1 - \begin{Bmatrix} k_d^2/2s^2 \\ k_s^2/2s^2 \end{Bmatrix} + O\left(\frac{p^4}{s^4}\right) \right]$$

It should be noted that since large $|s|$ here is necessarily on Br_s, (37) shows, assuming again that p is real, that α and β are real and large through the value of Im s. Note that for very large $|s|$, $\alpha = \beta = $Im s[10].

In accord with (35) and (36) the near field solution for $y=0$, and $y > 0$ differ [see remarks after (36)]. Because of the singular terms in (22), $y=h$ must also be given special consideration. Focusing first on the "interior" solution, $0 < y < h$, the contributions of $\tilde{u}_{cl}$ and $\tilde{v}_{cl}$ to the transformed long-time, near field solution are

$$\begin{Bmatrix} \tilde{u}_{cl} \\ \tilde{v}_{cl} \end{Bmatrix} = O\left[e^{is(h-y)}\Big/s^3 \right] \left[\sum_{n=1,3,5,..} (-)^{\frac{n-1}{2}} \frac{n\pi}{2h} a_n(p) \right.$$

$$\left. + \sum_{n=2,4,6,..} (-)^{\frac{n}{2}} \frac{n\pi}{2h} b_n(p) \right] \quad (38a)$$

[10]Since Im s has a change in sign from lower to upper half of Br_s, so do α and β. However, since $\tilde{u}$ and $\tilde{v}$ are even in α and β, it follows the large $|sh|$ approximation holds over all of Br_s in the usual way. In the work that follows s is chosen on the upper half of Br_s.

which is based on the fact that $R \simeq (k_d{}^2 - k_s{}^2)s^2 e^{-2ish}/2$ for large $|sh|$. The corresponding terms $\tilde{u}_{p1}$ and $\tilde{v}_{p1}$ give

$$
\left.
\begin{aligned}
\tilde{u}_{p1} &= \left[-\frac{1}{s^2} + O\left(\frac{1}{s^4}\right) \right] \sum_{n=1,3,5,\ldots} a_n(p) \cos\frac{n\pi y}{2h} \\
&\quad + \left[O\left(\frac{1}{s^3}\right) \right] \sum_{n=2,4,6,\ldots} \frac{n\pi}{2h} b_n(p) \cos\frac{n\pi y}{2h} \\
\tilde{v}_{p1} &= \left[\frac{1}{s^2} + O\left(\frac{1}{s^4}\right) \right] \sum_{n=2,4,6,\ldots} b_n(p) \sin\frac{n\pi y}{2h} \\
&\quad + \left[O\left(\frac{1}{s^3}\right) \right] \sum_{n=1,3,5,\ldots} \frac{n\pi}{2h} a_n(p) \sin\frac{n\pi y}{2h}
\end{aligned}
\right\} \tag{38b}
$$

Now since S_α and C_α may be written in the form

$$
\left\{ \begin{aligned} S_\alpha \\ C_\alpha \end{aligned} \right\} = \int_0^h \left[\frac{e^{\alpha(h-y')} \mp e^{-\alpha(h-y')}}{2h(1-y'/h)^{1/4}} \right] dy' \ , \tag{39}
$$

for large α these become

$$
\left\{ \begin{aligned} S_\alpha \\ C_\alpha \end{aligned} \right\} = \frac{e^{\alpha h}}{2} \int_0^h \frac{e^{-\alpha y'} dy'}{h(1-y'/h)^{1/4}} + O(1) \tag{40}
$$

Since α is a real, positive, large parameter in (40), according to (37), this integral may be approximated by Watson's Lemma [14], with the result

$$
\left\{ \begin{aligned} S_\alpha \\ C_\alpha \end{aligned} \right\} = \frac{e^{\alpha h}}{2} \left[\frac{1}{\alpha h} + \frac{1}{4\alpha^2 h^2} + \frac{5}{16\alpha^3 h^3} + \frac{45}{64\alpha^4 h^4} + O\left(\frac{1}{s^5}\right) \right] \tag{41}
$$

with an equivalent set for S_β and C_β. (41) may now be used to approximate $\tilde{u}_{c2}$ and $\tilde{v}_{c2}$ with the results

$$\tilde{u}_{c2} = -\frac{a_0(p)}{h}\left[\frac{1}{2}\left\{\frac{1}{s^2} - i\frac{\left(3k_d^2 - k_s^2\right)}{8hk_d^2 s^3} + O\left(\frac{1}{s^4}\right)\right\}\right.$$

$$\left. -\frac{C\left(k_d^2 - k_s^2\right)}{4k_s^2}\left\{\frac{i}{s^2} + \frac{1}{2hs^3} + O\left(\frac{1}{s^4}\right)\right\}\right]e^{-isy} \quad (42a)$$

$$\tilde{v}_{c2} = \frac{a_0(p)}{h}\left[\frac{C}{2}\left\{\frac{1}{s^2} + i\frac{\left(k_d^2 - 3k_s^2\right)}{8hk_s^2 s^3} + O\left(\frac{1}{s^4}\right)\right\}\right.$$

$$\left. -\frac{\left(k_d^2 - k_s^2\right)}{4k_d^2}\left\{\frac{i}{s^2} + \frac{1}{2hs^3} + O\left(\frac{1}{s^4}\right)\right\}\right]e^{-isy} \quad$$

A parts integration shows that the corresponding terms $\tilde{u}_{p2}$, and $\tilde{v}_{p2}$, to the same order of approximation, are

$$\tilde{u}_{p2} = -\frac{a_0(p)}{h}\left[\frac{1}{(1-y/h)^{1/4}s^2} + O\left(\frac{1}{s^3}\right) - \left\{\frac{1}{2}\left[\frac{1}{s^2} - i\frac{\left(3k_d^2 - k_s^2\right)}{8hk_d^2 s^3} + O\left(\frac{1}{s^4}\right)\right]\right.\right.$$

$$\left.\left. -\frac{C\left(k_d^2 - k_s^2\right)}{4k_s^2}\left[\frac{i}{s^2} + \frac{1}{2hs^3} + O\left(\frac{1}{s^4}\right)\right]\right\}e^{-isy}\right] \quad (42b)$$

$$\tilde{v}_{p2} = \frac{a_0(p)}{h}\left[C\left\{\frac{1}{(1-y/h)^{1/4}s^2} + O\left(\frac{1}{s^3}\right) +\right\} - \left\{\frac{C}{2}\left[\frac{1}{s^2} + i\frac{\left(k_d^2 - 3k_s^2\right)}{8hk_s^2 s^3} + O\left(\frac{1}{s^4}\right)\right]\right.\right.$$

$$\left.\left. -\frac{\left(k_d^2 - k_s^2\right)}{4k_d^2}\left[\frac{i}{s^2} + \frac{1}{2hs^3} + O\left(\frac{1}{s^4}\right)\right]\right\}e^{-isy}\right] \quad$$

Expanding $\tilde{u}_{c3}$ and $\tilde{v}_{c3}$ it is found that

$$\tilde{\tilde{u}}_{c3} = -\frac{Ca_0(p)}{h}\left[\left\{i\frac{\left(k_d^2-k_s^2\right)}{4k_s^2}\frac{1}{s^2} + O\left(\frac{p^2}{s^4}\right)\right\}e^{-isy}\right.$$

$$\left.-\frac{\left(3k_s^2-2k_d^2\right)e^{is(h-y)}}{2k^2\left(k_d^2-k_s^2\right)hs^3}\right]$$

$$\tilde{\tilde{v}}_{c3} = -\frac{Ca_0(p)}{h}\left[\left\{\frac{1}{2s^2} + O\left(\frac{p^2}{s^4}\right)\right\}e^{-isy}\right.$$

$$\left.+ i\frac{\left(3k_s^2-2k_d^2\right)e^{is(h-y)}}{2\left(k_d^2-k_s^2\right)s^3h}\right]$$

$$\text{(43a)}$$

$\tilde{\tilde{u}}_{p3}$ and $\tilde{\tilde{v}}_{p3}$ are given by

$$\tilde{\tilde{u}}_{p3} = \frac{Ca_0(p)\left(k_d^2-k_s^2\right)}{hk_s^2}\left[\frac{1}{s^3h} + O\left(\frac{p^2}{s^5}\right) + \left\{\frac{i}{4s^2} + O\left(\frac{p^2}{s^4}\right)\right\}e^{-isy}\right]$$

$$\tilde{\tilde{v}}_{p3} = -\frac{Ca_0(p)}{h}\left\{\left(1-\frac{y}{h}\right)\frac{1}{s^2} + O\left(\frac{p^2}{s^4}\right) - \left\{\frac{1}{2s^2} + O\left(\frac{p^2}{s^4}\right)\right\}e^{-isy}\right\}$$

$$\text{(43b)}$$

Since $\tilde{\tilde{u}}_{c4}$ and $\tilde{\tilde{v}}_{c4}$ contain the shift operator e^{-sa} they contribute nothing to the solution in $x < a$, and hence nothing to the near field solution.

From (38) it follows that

$$\tilde{\tilde{u}}_1 = -\frac{1}{s^2}\sum_{n=1,3,5,..} a_n(p)\cos\frac{n\pi y}{2h} + O\left(\frac{1}{s^3}\right)$$

$$\tilde{\tilde{v}}_1 = \frac{1}{s^2}\sum_{n=2,4,6,..} b_n(p)\sin\frac{n\pi y}{2h} + O\left(\frac{1}{s^3}\right)$$

$$\text{(44)}$$

and from $(42)^{11}$

$$\left.\begin{array}{l} \widetilde{u}_2 = -\dfrac{a_0(p)}{h(1-y/h)^{1/4}s^2} + O\!\left(\dfrac{1}{s^3}\right) \\[4mm] \widetilde{v}_2 = \dfrac{Ca_0(p)}{h(1-y/h)^{1/4}s^2} + O\!\left(\dfrac{1}{s^3}\right) \end{array}\right\} \qquad (45)$$

and from $(43)^{11}$

$$\left.\begin{array}{l} \widetilde{u}_3 = O\!\left(\dfrac{1}{s^3}\right) \\[4mm] \widetilde{v}_3 = -\dfrac{Ca_0(p)}{hs^2}\left(1-\dfrac{y}{h}\right) + O\,\dfrac{p^2}{s^4}^{12} \end{array}\right\} \qquad (46)$$

It follows from (44) to (46) that the time transformed long time-near field solution, valid for the plate interior, $0<y<h$, is given by

$$\left.\begin{array}{l} \bar{u}(x,y,p) \simeq -\left[\dfrac{a_0(p)}{h(1-y/h)^{1/4}} + \displaystyle\sum_{n=1,3,5,..} a_n(p)\cos\dfrac{n\pi y}{2h}\right]x \\[6mm] \bar{v}(x,y,p) \simeq \left[\dfrac{Ca_0(p)}{h}\left\{\dfrac{1}{(1-y/h)^{1/4}} - \left(1-\dfrac{y}{h}\right)\right\} + \displaystyle\sum_{n=2,4,6,..} b_n(p)\sin\dfrac{n\pi y}{2h}\right]x \end{array}\right\} \qquad (47)$$

The corresponding solution for $y=0$, is written by first noting that $\widetilde{u}_p$, $\widetilde{v}_p$, and $\widetilde{v}_c$ are all zero, and hence both $\widetilde{u}_{ci}^R$ and $\widetilde{u}_{ci}^0$ must be evaluated in (35) to calculate $\widetilde{\bar{u}}(s,0,p) = \widetilde{\bar{u}}_c(s,0,p)$. It is found by approximating $\widetilde{u}_{ci}^0$ in (35), and adding it to $\widetilde{u}_{ci}$ in (38a), evaluated at $y=0$, that

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[11] where cancellation of e^{-isy} terms occurred to the order indicated. From the pattern found in these, it is reasonable to expect that the same pattern of cancellation would result to any arbitrary higher order.

[12] When $\widetilde{v}_3$ is used in connection with the $y=h$ solution, this term is replaced by $O(1/s^3)$.

$$\tilde{u}_1(s,0,p) = \tilde{u}_{c1}(s,0,p) = -\frac{1}{s^2} \sum_{n=1,3,5,..} a_n(p) + O\left(\frac{1}{s^3}\right) \tag{48}$$

Recalling that $\tilde{u}_{c2}$ has both super R and zero contributions in it, the first of (42a), evaluated at $y=0$, gives

$$\tilde{u}_2(s,0,p) = \tilde{u}_{c2}(s,0,p) = \left[-\frac{a_0(p)}{h}\frac{1}{s^2}\left\{1-i\frac{C\left(k_d^2-k_s^2\right)}{2k_s^2}\right\} + O\left(\frac{1}{s^3}\right)\right] \tag{49}[13]$$

And by approximating $\tilde{u}_{c3}^0$ in (35), and adding it to $\tilde{u}_{c3}$ in (43a), evaluated at $y=0$, there results

$$\tilde{u}_3(s,0,p) = \tilde{u}_{c3}(s,0,p) = -\frac{Ca_0(p)}{h}\left[i\frac{\left(k_d^2-k_s^2\right)}{2k_s^2 s^2} + O\left(\frac{1}{s^3}\right)\right] \tag{50}$$

Adding (48), (49), and (50), and again noting that $\tilde{\tilde{v}}=0$, the time transformed, long time-near field solution for the plate mid-plane, $y=0$, is

$$\left.\begin{array}{l} \bar{u}(x,y,p) \simeq -\left[\dfrac{a_0(p)}{h} + \displaystyle\sum_{n=1,3,5,..} a_n(p)\right]x \\[12pt] \bar{v}(x,y,p) = 0 \end{array}\right\} \tag{51}$$

It may be noted that (51) can be obtained from (47) by letting $y \to 0$. One would expect this from continuity arguments. However, the process used here to derive (47) and (51) is instructive in showing that (47) is contributed by the particular integrals of $\tilde{u}$ and $\tilde{v}$, whereas the first of (51) comes from the complementary function.

The near field solution is not singular when $y=h$. This is clear from (35) and (36), if it is noted that $\tilde{u}_{p2}$ and $\tilde{v}_{p2}$ involve integrable singularities only. This means that the singular terms in (47) are not present in the

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[13]. The additional 1/2 in these terms stems from e^{isy} in $\cosh \alpha y$ which survives for $y=0$ only.

solution for y=h.   Now since all other terms in (47) vanish at y=h, the near field solution now comes from the order terms in (44) and (46)[14], i. e.

$$\left\{\begin{array}{c} \tilde{\bar{u}}(s, h, p) \\ \\ \tilde{\bar{v}}(s, h, p) \end{array}\right\} = O\left(\frac{1}{s^3}\right) \tag{52}$$

which after inversion yields the behavior

$$\left\{\begin{array}{c} \bar{u}(x, h, p) \\ \\ \bar{v}(x, h, p) \end{array}\right\} = O(x^2) \tag{53}$$

The actual terms corresponding to (52) [and (53)] are easily derived. Without the singular terms in (47) the remaining terms there behave as $(1-y/h)x$ near y=h.   So if $(1-y/h)$ gets small at the same rate as x does, this behavior is in agreement with that in (53).   The singular terms in (47), of course, get large as $y \rightarrow h$ which is not consistent with their absence at y=h.   This suggests that additional terms of the singular type in (47) might be involved, all of which sum to give zero.   Indeed carrying the parts integration of $\tilde{\bar{u}}_{p2}$ on further, for example ($\tilde{\bar{v}}_{p2}$ is similar), shows that for small x it has the singular terms

$$\bar{u}_{p2}(x, y, p) \simeq -\frac{a_0(p)}{h}\left\{\left[\frac{x}{(1-y/h)^{1/4}} + \frac{5(k^2-1)x^3}{96h^2(1-y/h)^{9/4}} - \frac{65(2k^2-3)x^5}{512h^4(1-y/h)^{17/4}} + \right.\right.$$
$$\left.\left. + \frac{C(k^2-1)}{k^2}\left[\frac{x^2}{8h(1-y/h)^{5/4}} - \frac{15x^4}{256h^3(1-y/h)^{13/4}} + \right]\right\} \tag{54}$$

After the first term in (54) there is sign alternation of the terms in each of the brackets, and this pattern is true for the higher order terms also.

_____

[14] The order term in (45) is not present in the y=h solution.

Hence, it is reasonable to expect, that as $y \to h$, the sum of the terms in (54) will approach zero. It also follows then, for the leading singular term in (54) to represent $\bar{u}_{p2}$ in (47), as it does, x must vanish at a faster rate than $(1-y/h)^{1/4}$ does, i.e. the near field solution (47) is limited to smaller and smaller x values as the corner at $y=h$ is approached. Otherwise the use of the additional terms in (54) will be needed, which obviously will still have limitations as y gets closer and closer to h. One can, of course, evaluate at a station quite near $y=h$ in this manner, then evaluate at $y=h$ from the terms of which give (53).

It is clear then, that once the unknowns $a_0(p)$, $a_n(p)$, and $b_n(p)$ are numerically evaluated, a simple inversion of (47) [since it has already been shown the inverses of the unknowns behave as $H(t)$] will yield the long time-near field solution for the present problem.

Verification of Long-Time Solution.

Short of writing the inverse (26), which will be postponed until the numerical work for the coefficients of the edge unknowns is carried out (as discussed earlier), some things can be done on verifying that (15), defined by (35) and (36), will yield a solution satisfying the present boundary value problem, i.e. the differential equations (1), the initial conditions (3), the boundary conditions at the edge $x=0$, (21), the plate face $y=h$, and center $y=0$, the inverses of (10) and (11), and the conditions at $x \to \infty$, (4).

Verification that the long-time solution, once it is written, satisfies the differential equations (1) can be shown by direct substitution. Since the problem is based on (1), a system of hyperbolic equations, one would expect the general solution (15), defined by (12) and (8), to have a singularity

free half plane $\operatorname{Re} p \geq \gamma$, a positive constant. Hence completion of Br_p in this domain would give the corresponding solution as

$$\left\{\begin{matrix} u(x, y, t) \\ v(x, y, t) \end{matrix}\right\} = 0 \ , \qquad \text{for } 0 < t < \sqrt{x^2 + y^2}/c_d \tag{55}$$

hence satisfying not only the initial conditions (3) on the displacements, but also on the velocities there.

Direct verification of the edge boundary conditions (21) can be obtained by setting $x=0$ in (47) and (53). A differentiation of these w. r. t. x, and then letting $x \to 0$ [noting the order terms in (44) to (46), and the higher order x terms in (54)], recovers the assumed expressions for the strains $\bar{u}_x(0, y, p)$ and $\bar{v}_x(0, y, p)$ in (22). Note, in the case of (54), after this differentiation, that setting $x=0$ away from $y=h$, leaves only the leading term as it should be. But when $y \to h$, it is now reasonable to expect that the terms only after the first (which in each of the brackets alternate) would sum to zero.

Direct substitution of $y=0$ in the second of (13), with the second of (35) and (36) defining the integrand there, gives $\bar{v}(x, 0, p)=0$, hence from (15), $v(x, 0, t)=0$ (independent of time, hence giving the same result from (26) for long time). Thus the double inverse of the second of (11) is satisfied. Now, using (13), with (35) and (36), to form $\bar{\sigma}_{xy}(x, y, p)/\mu$, in accord with the third of (2), it is easily shown by direct substitution of $y=0$ that $\bar{\sigma}_{xy}(x, 0, p)$ and therefore $\sigma_{xy}(x, 0, t)$ are zero   Thus the double inverse of the first of (11) is also satisfied.

Now substituting $y=h$ directly into the Br_s integral just formed for $\bar{\sigma}_{xy}(x, y, p)/\mu$, and a similarly formed one for $\bar{\sigma}_y(x, y, p)/(\lambda+2\mu)$, in accord

with the second of (2), it is easily shown that $\overline{\sigma}_{xy}(x, h, p)/\mu=0$, and $\overline{\sigma}_{y}(x, h, p)/(\lambda+2\mu)=\sigma_0\delta(x-a)\overline{G}(p)/(\lambda+2\mu)$, and hence $\sigma_{xy}(x, h, t)/\mu=0$, and $\sigma_{y}(x, h, t)=\sigma_0\delta(x-a)G(t)$. Thus the double inverses of (10) [see also (20)a] are satisfied.

It remains to show the conditions at $x \rightarrow \infty$, (4), are satisfied. In the case of the general solution (15), with (35) and (36), again (55) assures satisfaction of (4) in advance of the wave front, i. e. for $\sqrt{x^2+y^2} > c_d t$. However, back of this wave front, and for the harmonic problem solution, (13), with (35) and (36), it is necessary to show these solutions do not become unbounded. The Tauberian theorem, in which $s \rightarrow 0$, can be used to show that this far field condition is satisfied. There are two cases to be considered. First, $s \rightarrow 0$ and p small but not zero, and second, s and $p \rightarrow 0$ together. Approximations of (35) and (36) corresponding to the first case show, at most, contributions of $O(1)$ in s for $0 \leq y \leq h$. It follows that $u(x, y, t)$ and $v(x, y, t)$ behave, at most, as $\delta(x)$ for $x \rightarrow \infty$, and therefore vanish. Since $\tilde{u}_x$, $\tilde{v}_x$, and higher order derivatives in x, involve higher order behaviors in s (as $s \rightarrow 0$), it follows their inverses are also zero for $x \rightarrow \infty$. $\tilde{u}_y$, $\tilde{v}_y$, and higher order derivatives in y, differ from $\tilde{u}$ and $\tilde{v}$ only through higher powers of α and β, and since for $s \rightarrow 0$ the latter behave as k_d and k_s, respectively, the inverses of these derivatives behave like u and v, vanishing as $x \rightarrow \infty$. It follows that for the present case (4) is satisfied everywhere for large x.

Now for the case of s and $p \rightarrow 0$ together, R behaves as

$$R \simeq -b^2 h\beta k_s^{\ 2}\left[s^2-k_p^{\ 2}\right] \tag{56}$$

where $b^2 = c_p^2/c_s^2$, $k_p^2 = p^2/c_p^2$, c_p being the "plate velocity" $\sqrt{E/\rho(1-\nu^2)}$. (56) is well known, the zeros of $\left(s^2 - k_p^2\right)$ therein corresponding to the low frequency-long wave approximation to the lowest mode of R in (14), i.e., the plane stress approximation. Approximating (35) and (36) for this case, shows that for $\widetilde{u}(s, y, p)$

$$\widetilde{u}_{c1}^{R}(s, y, p) \simeq - \frac{k^2}{b^2\left(s^2 - k_p^2\right)} \sum_{n=1,3,5,..} (-)^{\frac{n-1}{2}} \frac{2}{n\pi} a_n(p) \tag{57}$$

$$\widetilde{u}_{c2}(s, y, p) \simeq - \frac{k^2 X a_0(p)}{hb^2\left(s^2 - k_p^2\right)} \tag{58}$$

survive for $0 \leq y \leq h$, where $X = C_{k_d} = C_{k_s}$. Now since (33) shows the coefficients $a_0(p)$ and $a_n(p)$ [and $b_n(p)$] behave as $1/p$, for $G(t) = H(t)$ the step input, simple table inversions of (13) and (15), with $\widetilde{u} \simeq \widetilde{u}_{c1}^{R} + \widetilde{u}_{c2}$ where the latter are given by (57) and (58), yield the behavior of $u(x, y, t)$ to be

$$u(x, y, t) \simeq U(t - x/c_p) \tag{59}$$

for the long time-far field solution[15], where U is a constant.

Similarly, for $\widetilde{v}(s, y, p)$ in $0 < y \leq h$, approximation shows that

$$\widetilde{v}_{c1}(s, y, p) \simeq \frac{(2-k^2) y s}{b^2\left(s^2 - k_p^2\right)} \sum_{n=1,3,5,..} (-)^{\frac{n-1}{2}} \frac{2}{n\pi} a_n(p) \tag{60}$$

$$\widetilde{v}_{c2}(s, y, p) \simeq \frac{(2-k^2) y s X a_0(p)}{b^2\left(s^2 - k_p^2\right)} \tag{61}$$

survive, which leads to the double inverse

$$v(x, y, t) \simeq V y H(t - x/c_p) \tag{62}$$

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[15] An equal term, but corresponding to a negative traveling wave, is ruled out in the inversion of (15), since it leads to an unbounded contribution.

for the long time-far field solution, where V is a constant. It should be noted that (59) and (62) are restricted to a small region just after $t=x/c_p$. There, however, they correspond to non-decaying, in space (x) and time, bounded disturbances. Because of the latter these solutions satisfy the requirement of boundedness in $\sqrt{x^2+y^2} <c_d t$, since $c_p<c_d$. The non-decaying nature of this disturbance has been observed before, in several related problems, e.g. the work of Curtis [15]. It is of interest to note that the time behaviors of u and v in (59) and (62), at a plate station, agree with the time behaviors of the corresponding radial and thickness displacements in the solution, given by the author [16], for the related problem of excitation of an elastic plate by symmetric normal point loads (see Figs. 4, 5 in [16]). The latter displacements, however, decay spatially. It is planned to include a more complete study of the lowest mode disturbance in the future work on the present problem.

§4. COMMENTS AND FUTURE WORK.

Plans to evaluate further the solution treated here, have already been discussed. It is of, at least, equal importance to remind the reader that solution of the integral equations (19), basic to this work, has so far been limited to the eigenfunction technique employed by Benthem in his problems from elastostatics [see (22)]. Their applicability here has been argued on the basis of the long time-low frequency analog. It is therefore desirable to look into other possible more general means of evaluating (19), so that, perhaps short time-high frequency phenomena can be investigated in the problem treated here.

Further, solution of (19) for problems of the other non-mixed edge type, i.e. free edge, including sudden edge loads, is also of strong interest. Some work has been done on these problems and plans are to continue it.

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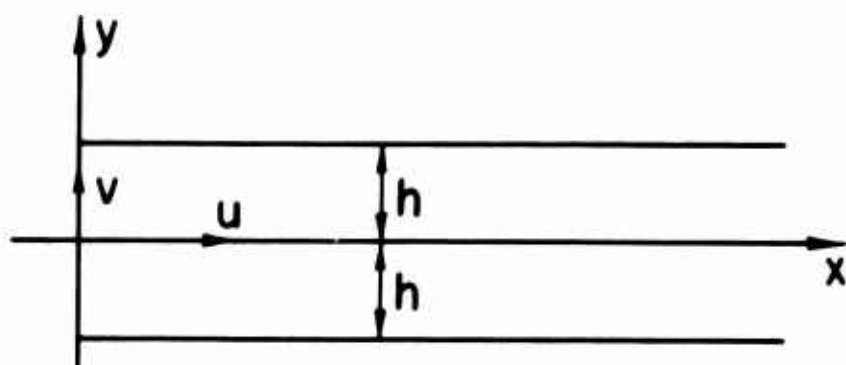
Fig. 1. Coordinates and Displacements for the
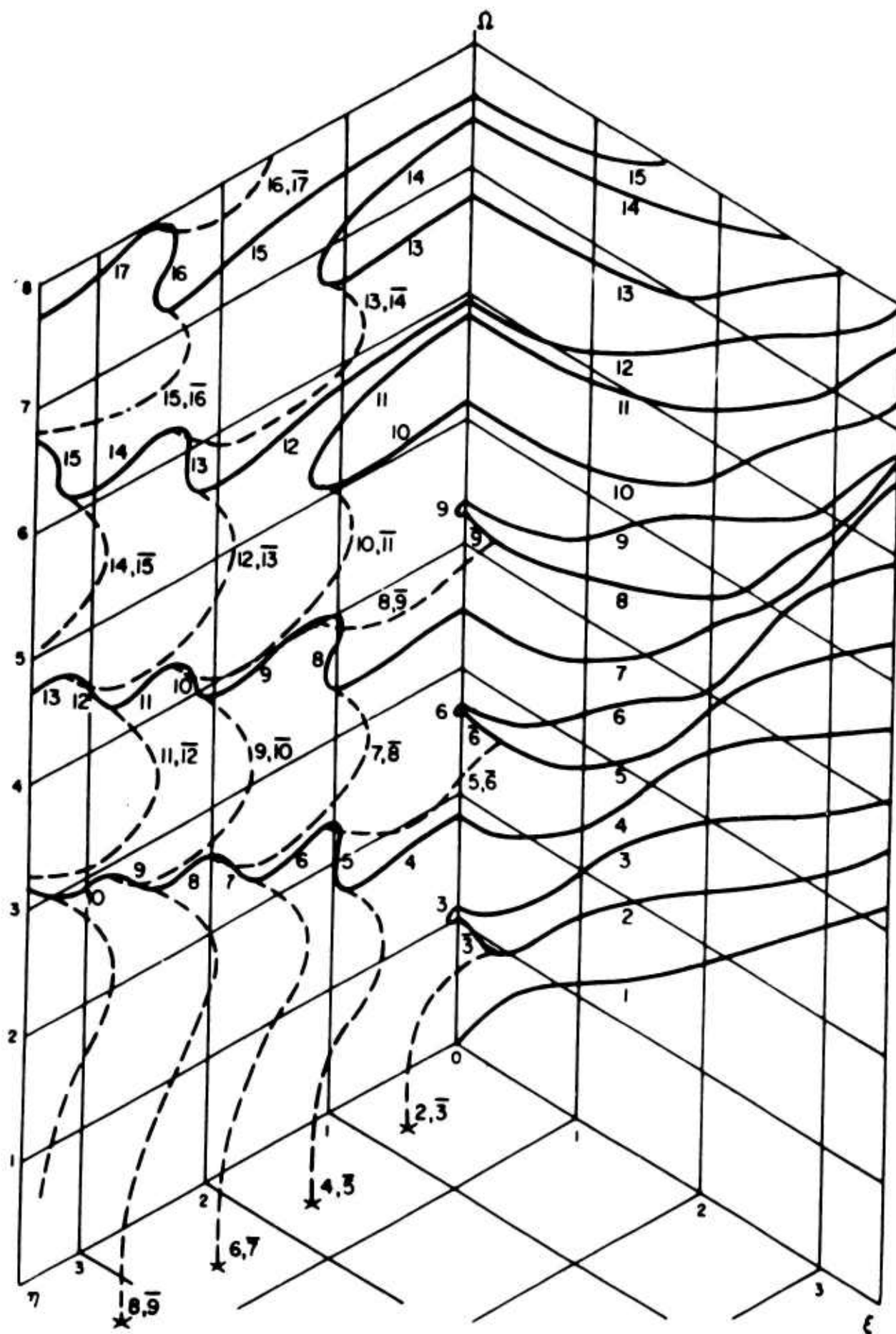Semi-Infinite Plate In Plane Strain.

Fig. 2.   Frequency Spectra for Axially Symmetric Waves in an Infinite
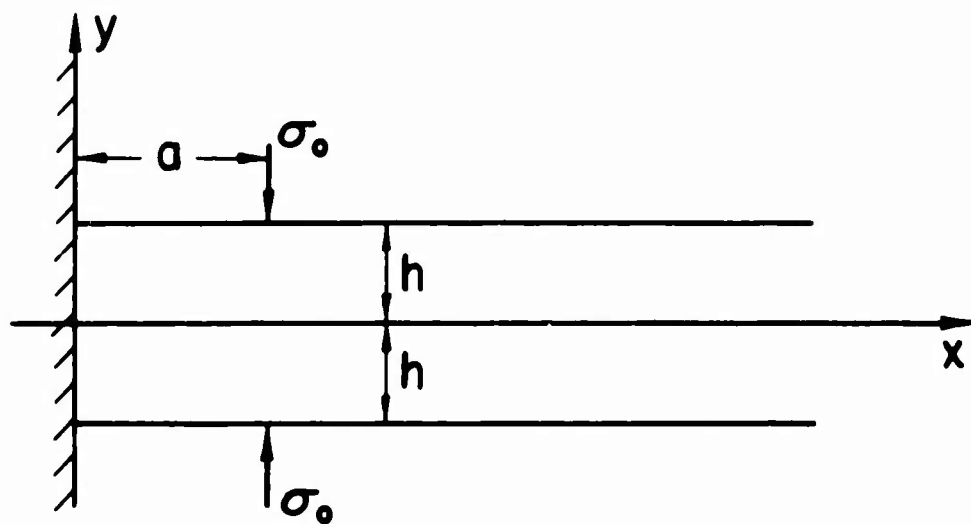Circular Cylindrical Rod (from Onoe, McNiven, and Mindlin [5]).

Fig. 3. Plate with Built-In-Edge Excited by Face
Symmetric Normal Line Loads.

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13. ABSTRACT

Within the framework of the "exact" linear theory an important class of wave propagation problems in elastic waveguides, involving non-mixed edge conditions (like stress or displacement), have remained unsolved. Basically, this is because known separation methods (classical or integral transforms) do not "ask" in a natural way for the given edge information. A means for solving some problems in this class, focused on the semi-infinite plate, as an example, is presented here. In the method a Laplace transform is used on the propagation coordinate, say x. Exploitation of the boundedness condition on the solution, at $x \to \infty$, generates two coupled integral equations for the edge unknowns (displacements and strains), which depend, parametrically, on those complex wave number roots of the governing Rayleigh-Lamb frequency equation representing unbounded waves. Solution of these equations determines the transformed solution of the problem, which can be inverted through known techniques. Excitation of a plate with a built-in edge is treated as an example.

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